JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 77, 381–402 (1980)

On a Class of Similarity Solutions of the Porous Media Equation, III

B. H. GILDING

Mathematisch Instituut, Rijksuniversiteit te Leiden, Leiden, The Netherlands

Submitted by W. F. Ames

1. INTRODUCTION

In two recent papers [11, 12] the present author and L. A. Peletier investigated a class of similarity solutions of the porous media equation. A partial characterization of these solutions was given. In the present paper we tie up the loose ends.

The porous media equation

$$u_t = (u^m)_{xx}, \qquad m > 1, \tag{1}$$

is so called due to its description of the flow of a polytropic gas in a homogeneous porous media. The equation has further applications in the theories of diffusion, heat conduction, boundary layers, and solar prominences [2].

Equation (1) is parabolic at any point (x, t), where $u(x, t) > 0$, but at points where $u(x, t) = 0$ it is not. As a result of this degeneracy (1) need not always have a classical solution. However, classes of weak solutions of the Cauchy problem and the Cauchy–Dirichlet problem have been defined by Oleinik *et al.* [17] and existence and uniqueness theorems proved within these classes.

We shall study (1) in the half-strip $S_T = (0, \infty) \times (0, T]$, where T is positive. As in [11, 12], we shall be concerned with similarity solutions of (1) in S_T of the following three types:

I. $u_1(x, t) = (t + \tau)^\alpha f_1(\eta)$, $\eta = x(t+\tau)^{-\{1+(m-1)\alpha\}/2}$ for $\tau > 0$;

II. $u_2(x, t) = (\tau - t)^\alpha f_2(\eta)$, $\eta = x(\tau - t)^{-\{1+(m-1)\alpha\}/2}$ for $\tau > T$;

III. $u_3(x, t) = e^{\alpha(t+\tau)} f_3(\eta)$, $\eta = x \exp\{-\frac{1}{2}\alpha(m-1)(t+\tau)\}$ for any τ.

Substitution of u_1, u_2, and u_3 into (1) leads respectively to the following equations for the functions f_1, f_2, and f_3:

I. $(f_1^m)'' + \frac{1}{2}\{1 + (m-1)\alpha\}\, \eta f_1' = \alpha f_1, \qquad 0 < \eta < \infty, \qquad$ (2a)

0022-247X/80/100329–15$02.00/0

$$\text{II.}\quad (f_2^m)'' - \tfrac{1}{2}\{1 + (m-1)\alpha\}\,\eta f_2' = -\alpha f_2, \qquad 0 < \eta < \infty, \tag{2b}$$

$$\text{III.}\quad (f_3^m)'' + \tfrac{1}{2}\alpha(m-1)\,\eta f_3' = \alpha f_3, \qquad 0 < \eta < \infty. \tag{2c}$$

These similarity transformations have been investigated by Barenblatt [5–7] and Marshak [16], and a number of explicit solutions, summarized in [11], have been found by various authors [1, 5, 6, 8, 14, 17, 18, 24, 25]. Nonetheless, it is only recently that a rigorous study of the similarity solutions has begun. This has centred on the equation

$$(k(f)\,f')' + \tfrac{1}{2}\eta f' = 0, \qquad 0 < \eta < \infty, \tag{3}$$

where $k(s)$ is defined real and continuous for $s \geqslant 0$, with $k(0) = 0$ and $k(s) > 0$ if $s > 0$, of which (2a) for $\alpha = 0$ is a special case. Equation (3) has been studied by Atkinson and Peletier [3, 4] and by Shampine [20–22]; under more restrictive conditions on the function k by Craven and Peletier [9], by Lee [15], and by Tam [23]; and on the interval $-\infty < \eta < \infty$ by van Duyn and Peletier [10].

In this paper we consider the equation

$$(f^m)'' + p\eta f' = qf, \qquad 0 < \eta < \infty, \tag{4}$$

in which p and q are real constants. Plainly (4) encompasses (2a)–(2c).

In view of the properties of (1) it is necessary to define a weak solution of (4). A function f is said to be a weak solution of (4) if (i) f is bounded nonnegative and continuous on $[0, \infty)$, (ii) $(f^m)(\eta)$ has a continuous derivative with respect to η on $(0, \infty)$, and (iii) f satisfies the identity

$$\int_0^\infty \phi'\{(f^m)' + p\eta f\}\,d\eta + (p+q)\int_0^\infty \phi f\,d\eta = 0$$

for all $\phi \in C_0^1(0, \infty)$.

It was shown in [11, 12] that any nontrivial weak solution of (4) $f(\eta)$ (i.e., any weak solution other than the identically zero solution) must be of one of the following two forms:

TYPE A. *There exist parameters* $U \geqslant 0$ *and* β, *with* $\beta > 0$ *if* $U = 0$, *and a constant* $a = a(U, \beta) \in (0, \infty)$ *such that* f *is a positive classical solution of* (4) *on the interval* $(0, a)$ *satisfying the boundary conditions*

$$\begin{aligned} f(0) &= U, & (f^m)'(0) &= \beta, \\ f(a) &= 0, & (f^m)'(a) &= 0, \end{aligned} \tag{5}$$

and

$$f(\eta) \equiv 0 \qquad \text{on} \quad [a, \infty).$$

TYPE B. *There exist parameters* $U \geqslant 0$ *and* β, *with* $\beta > 0$ *if* $U = 0$, *such that* f *is a bounded positive classical solution of problem* (4), (5) *on* $(0, \infty)$.

We shall say that the trivial solution is of Type A with $U = \beta = 0$.

For fixed p and q let $\mathscr{S}_A$ denote the set of values (U, β) for which there exists a solution of Type A and let $\mathscr{S}_B$ denote the set of values (U, β) for which there exists a solution of Type B. Then the following results have been established [11, 12]:

(a) If $q < 0$ and $2p + q < 0$ then $\mathscr{S}_A = \{(0, 0)\}$ and $\mathscr{S}_B = \varnothing$.

(b) If $q < 0$ and $2p + q = 0$ then $\mathscr{S}_A = \{(0, \beta)\colon\ 0 \leqslant \beta < \infty\}$ and $\mathscr{S}_B = \varnothing$.

(c) If $q \leqslant 0$ and $2p + q > 0$ then given any $U \geqslant 0$ there exists a unique $\beta^* = \beta^*(U)$ such that $\mathscr{S}_A = \{(U, \beta^*(U))\colon 0 \leqslant U < \infty\}$ and $\mathscr{S}_B = \{(U, \beta)\colon 0 \leqslant U < \infty \text{ and } \beta^*(U) < \beta < \infty\}$.

(d) If $q > 0$ and $p \geqslant 0$ then given any $U \geqslant 0$ there exists a unique $\beta^* = \beta^*(U)$ such that $\mathscr{S}_A = \{(U, \beta^*(U))\colon\ 0 \leqslant U < \infty\}$ and $\mathscr{S}_B = \varnothing$.

(e) If $q > 0$ and $p < 0$, or $q = 0$ and $p \leqslant 0$, then $\mathscr{S}_A = \{(0, 0)\}$ and $\mathscr{S}_B \subset (0, \infty) \times (-\infty, 0]$, moreover, given any $U > 0$ there exists at least one $\beta \in (-\infty, 0]$ such that $(U, \beta) \in \mathscr{S}_B$. Furthermore, if $q = 0$ or $p + q \geqslant 0$ given any $U > 0$ there exists at most one $\beta \in (-\infty, 0]$ such that $(U, \beta) \in \mathscr{S}_B$.

Finally, given any $(U, \beta) \in \mathscr{S}_A \cup \mathscr{S}_B$ there exists at most one weak solution of (4) satisfying (5).

In the present paper we shall settle the question of uniqueness with respect to U of solutions of Type B for the remaining cases of class (e) above, and investigate the asymptotic behaviour as $\eta \to \infty$ of solutions of Type B.

The paper is structured as follows. For convenience, in the next section, we shall just list a number of basic results to which frequent reference will be made in the course of the present analysis. Then, in Section 3, we shall settle the open uniqueness question and succinctly summarize the existence and uniqueness results for all values of p and q. In the following section, Section 4, we shall identify the asymptotic behaviour as $\eta \to \infty$ of solutions of Type B and discuss the rate at which these solutions approach the asymptotic limit. Finally, in Section 5 we shall investigate the dependence of the asymptotic limit as $\eta \to \infty$ of a solution of Type B on the parameters U and β in the initial condition (5).

2. FUNDAMENTAL RESULTS

LEMMA 1 (On the classical solution of (4)). *Let* $(\eta_0, U_0, \beta_0) \in \Omega = [0, \infty) \times (0, \infty) \times (-\infty, \infty) \cup \{(0, 0, \beta)\colon 0 < \beta < \infty\}$; *then there exists an*

$\varepsilon > 0$ *such that* (4) *has a unique positive classical solution in* $\mathscr{B}_\varepsilon(\eta_0) = \{\eta \in (0, \infty) : |\eta - \eta_0| < \varepsilon\}$ *satisfying*

$$f(\eta_0) = U_0, \qquad (f^m)'(\eta_0) = \beta_0. \tag{6}$$

Furthermore, positive classical solutions of problem (4), (6) *depend continuously on initial data in* Ω.

This lemma is a simple consequence of the standard theory of ordinary differential equations [13] since for any $(\eta_0, U_0, \beta_0) \in \Omega$ problem (4), (6) may be *locally* transformed to an ordinary differential equation for which the standard theory applies; cf. [11, 12].

LEMMA 2 (Properties of solutions of Type A satisfying $f(0) = U > 0$ [11]). *Suppose* $p \geqslant 0$ *and* $2p + q > 0$. *Then given any* $U > 0$ *there exists a unique solution of Type* A, $f(\eta)$, *satisfying* $f(0) = U$. *Furthermore, setting* $a = \sup\{\eta \in [0, \infty) : f(\eta) > 0\}$:

(i) *if* $p + q < 0$ *there exists a point* $\bar{\eta} \in (0, a)$ *such that* $(f^m)'(\eta) > 0$ *for all* $\eta \in [0, \bar{\eta})$, $(f^m)'(\bar{\eta}) = 0$ *and* $(f^m)'(\eta) < 0$ *for all* $\eta \in (\bar{\eta}, a)$,

(ii) *if* $p + q = 0$, $(f^m)'(0) = 0$ *and* $(f^m)'(\eta) < 0$ *for all* $\eta \in (0, a)$,

(iii) *if* $p + q > 0$, $(f^m)'(\eta) < 0$ *for all* $\eta \in [0, a)$.

LEMMA 3 (Properties of solutions of Type B [12]). *Let* $f(\eta)$ *be a solution of Type* B.

(i) *If* $q < 0$ *and* $2p + q > 0$ *then either there exists a point* $\bar{\eta} \in (0, \infty)$ *such that* $(f^m)'(\eta) > 0$ *for all* $\eta \in [0, \bar{\eta})$, $(f^m)'(\bar{\eta}) = 0$ *and* $(f^m)'(\eta) < 0$ *for all* $\eta \in (\bar{\eta}, \infty)$; $(f^m)'(0) = 0$ *and* $(f^m)'(\eta) < 0$ *for all* $\eta \in (0, \infty)$; *or* $(f^m)'(\eta) < 0$ *for all* $\eta \in [0, \infty)$.

(ii) *If* $q = 0$ *and* $p > 0$ *then either* $(f^m)'(\eta) \neq 0$ *for all* $\eta \in [0, \infty)$, *or* $f(\eta) \equiv U$ *on* $[0, \infty)$.

(iii) *If* $q > 0$ *and* $p < 0$ *then* $(f^m)'(\eta) < 0$ *for all* $\eta \in [0, \infty)$.

(iv) *If* $q = 0$ *and* $p \leqslant 0$ *then* $f(\eta) \equiv U$ *on* $[0, \infty)$.

Moreover,

$$qf(\eta) \to 0, \qquad (f^m)'(\eta) \to 0 \qquad \textit{as} \quad \eta \to \infty. \tag{7}$$

3. UNIQUENESS OF SOLUTIONS OF TYPE B FOR $q > 0$ AND $p < 0$, OR, $q = 0$ AND $p \leqslant 0$

Suppose $U > 0$. Then for $q = 0$ it was established in [12] that there exists one and only one solution of Type B satisfying $f(0) = U$, namely, $f(\eta) \equiv U$.

On the other hand, for $q > 0$ it was shown that there exists at least one such solution, and furthermore that if $p + q \geqslant 0$ this solution is unique.

In this section we shall show that for the remaining values of p and q there is also at most one bounded positive classical solution of (4) on $(0, \infty)$ satisfying $f(0) = U$. We begin with the following lemma.

LEMMA 4. *Assume $q > 0$ and $p < 0$, and let f_1 and f_2 be two monotonic decreasing solutions of* (4) *in an interval* $(0, a)$, $0 < a < \infty$, *satisfying the initial condition*

$$f_i(0) = U > 0, \qquad (f_i^m)'(0) = \beta_i < 0,$$

for $i = 1, 2$. Then if $\beta_1 > \beta_2$ it holds that

$$f_1'(\eta) > f_2'(\eta) \qquad \textit{for all} \quad \eta \in [0, a).$$

Proof. Suppose that there exists a point $\eta_0 \in (0, a)$ such that $f_1'(\eta_0) \leqslant f_2'(\eta_0)$. Then since

$$f_1'(0) = \frac{1}{m} U^{1-m}\beta_1 > \frac{1}{m} U^{1-m}\beta_2 = f_2'(0),$$

by continuity we can define a point $\eta_1 \in (0, \eta_0]$ such that

$$f_1'(\eta) > f_2'(\eta) \qquad \text{for all} \quad \eta \in [0, \eta_1)$$

and

$$f_1'(\eta_1) = f_2'(\eta_1). \tag{8}$$

Plainly then

$$f_1(\eta) > f_2(\eta) \qquad \text{for all} \quad \eta \in (0, \eta_1]. \tag{9}$$

Hence, substituting f_1 and f_2 into (4), it holds that

$$(f_1^m - f_2^m)''(\eta) = -p\eta(f_1 - f_2)'(\eta) + q(f_1 - f_2)(\eta) > 0$$

for all $\eta \in (0, \eta_1]$. Whence

$$(f_1^m - f_2^m)'(\eta_1) > \beta_1 - \beta_2 > 0.$$

However, since f_2 is monotonic decreasing, by (8) and (9) we have

$$\begin{aligned}(f_1^m)'(\eta_1) &= m f_1^{m-1}(\eta_1) f_1'(\eta_1) = m f_1^{m-1}(\eta_1) f_2'(\eta_1) \\ &\leqslant m f_2^{m-1}(\eta_1) f_2'(\eta_1) = (f_2^m)'(\eta_1),\end{aligned}$$

which is a contradiction.

To prove that when $q > 0$ and $p < 0$ there exists at most one solution of Type B satisfying $f(0) = U$ is now straightforward.

Suppose that there exist two such solutions f_1 and f_2. Then by Lemma 3, f_1 and f_2 are strictly monotonic decreasing on $[0, \infty)$. Without any loss of generality suppose that $0 > (f_1^m)'(0) > (f_2^m)'(0)$. Then by Lemma 4, $f_1'(\eta) > f_2'(\eta)$ for all $\eta \in [0, \infty)$. It follows that

$$f_1(\eta) - f_2(\eta) > f_1(\eta_0) - f_2(\eta_0) > 0$$

for all $\eta > \eta_0 > 0$. However, this contradicts (7). We conclude that there exists at most one solution of Type B satisfying $f(0) = U$.

Compiling the results for this particular class of values of p and q, we have thus proved:

PROPOSITION 1. *Assume $q > 0$ and $p < 0$, or $q = 0$ and $p \leqslant 0$. Then given any $U > 0$ there exists precisely one solution of Type* B *satisfying $f(0) = U$.*

In the Introduction the existence and uniqueness results for weak solutions of (4), proved in [11, 12], were summarized. Proposition 1 strengthens these results for the case $q > 0$ and $p < 0$. However, by using the following observation the results may be strengthened still further.

Remark. Let $f(\eta)$ denote a weak solution of (4) satisfying (5). Then for any $\mu > 0$

$$h(\eta) = \mu^{-2/(m-1)} f(\mu\eta)$$

is also a weak solution of (4) with the initial condition

$$h(0) = \mu^{-2/(m-1)} U, \qquad (h^m)'(0) = \mu^{-(m+1)/(m-1)} \beta.$$

In view of this remark and Lemmas 2 and 3, the existence and uniqueness results may be compiled as follows.

THEOREM 1. *For fixed p and q let $\mathscr{S}_A$ denote the set of values (U, β) for which there exists a solution of Type* A *and let $\mathscr{S}_B$ denote the set of values (U, β) for which there exists a solution of Type* B.

(a) *If $q < 0$ and $2p + q < 0$ then $\mathscr{S}_A = \{(0,0)\}$ and $\mathscr{S}_B = \varnothing$,*

(b) *If $q < 0$ and $2p + q = 0$ then $\mathscr{S}_A = \{(0, \beta): 0 \leqslant \beta < \infty\}$ and $\mathscr{S}_B = \varnothing$.*

(c) *If $q \leqslant 0$ and $2p + q > 0$ then there exists a unique β_1 such that $\mathscr{S}_A = \{(U, U^{(m+1)/2}\beta_1): 0 \leqslant U < \infty\}$ and $\mathscr{S}_B = \{(U, \beta): 0 \leqslant U < \infty$ and $U^{(m+1)/2}\beta_1 < \beta < \infty\}$, where $\beta_1 > 0$ if $p + q < 0$, $\beta_1 = 0$ if $p + q = 0$ and $\beta_1 < 0$ if $p + q > 0$.*

(d) *If* $q > 0$ *and* $p \geqslant 0$ *then there exists a unique* $\beta_1 < 0$ *such that* $\mathscr{S}_A = \{(U, U^{(m+1)/2}\beta_1): 0 \leqslant U < \infty\}$ *and* $\mathscr{S}_B = \varnothing$.

(e) *If* $q > 0$ *and* $p < 0$, *or* $q = 0$ *and* $p \leqslant 0$, *then* $\mathscr{S}_A = \{(0, 0)\}$ *and there exists a unique* β_1 *such that* $\mathscr{S}_B = \{(U, U^{(m+1)/2}\beta_1): 0 < U < \infty\}$, *where* $\beta_1 < 0$ *if* $q > 0$ *and* $\beta_1 = 0$ *if* $q = 0$.

Moreover, for each $(U, \beta) \in \mathscr{S}_A \cup \mathscr{S}_B$ *there exists at most one weak solution of* (4) *satisfying* (5).

4. Asymptotic Behaviour of Solutions of Type B

Let $f(\eta)$ be a solution of Type B. Then it was shown in [12] that if $q = 0$, $f(\eta) \to V$ as $\eta \to \infty$ for some $V > 0$. In this section we shall extend this result to all values of p and q. To be precise we shall verify the following assertion.

Theorem 2. *Let* $f(\eta)$ *be a solution of Type* B. *Then*

$$f(\eta) \sim V\eta^{-\lambda} \qquad as \quad \eta \to \infty \tag{10}$$

for some $V > 0$, *where*

$$\lambda = -q/p \qquad for \quad p \neq 0$$
$$= 0 \qquad for \quad p = 0.$$

We shall prove the result in a number of stages. Moreover, since it has already been established for the case $q = 0$ we shall confine our attention to the case $q \neq 0$. Thus, by Theorem 1, we shall consider $q < 0$ and $2p + q > 0$, and $q > 0$ and $p < 0$. Note that in both these cases, $\lambda > 0$ implicitly.

To prove Theorem 2 we shall utilize properties of the auxiliary function defined by

$$\theta(\eta) = \eta^{\lambda} f(\eta).$$

Since $\lambda > 0$, immediately we have $\theta(0) = 0$ and $\theta(\eta) > 0$ for all $\eta \in (0, \infty)$. Moreover, substituting θ in (4) it holds that

$$(f^m)'' + p\eta^{1-\lambda}\theta' = 0 \qquad \text{for} \quad 0 < \eta < \infty, \tag{11}$$

and

$$(\theta^m)'' - 2m\lambda\eta^{-1}(\theta^m)' + p\eta^{1+(m-1)\lambda}\theta'$$
$$= -m\lambda(m\lambda + 1)\,\eta^{-2}\theta^m \qquad \text{for} \quad 0 < \eta < \infty. \tag{12}$$

A more significant property of the function θ is contained in the following lemma.

LEMMA 5. (i) *If $q < 0$ and $2p + q > 0$ there exists a point $\bar{\eta} \in (0, \infty)$ such that $\theta'(\eta) > 0$ for all $\eta \in (0, \bar{\eta})$, $\theta'(\bar{\eta}) = 0$ and $\theta'(\eta) < 0$ for all $\eta \in (\bar{\eta}, \infty)$.*

(ii) *If $q > 0$ and $p < 0$ then $\theta'(\eta) > 0$ for all $\eta \in (0, \infty)$.*

Proof. Suppose that $\theta'(\eta_0) = 0$ for some point $\eta_0 \in (0, \infty)$. Then by (12)

$$(\theta^m)''(\eta_0) = -m\lambda(m\lambda + 1)\,\eta_0^{-2}\theta^m(\eta_0) < 0.$$

Hence either $\theta'(\eta) > 0$ for all $\eta \in (0, \infty)$, or there exists a point $\bar{\eta} \in (0, \infty)$ such that $\theta'(\eta) > 0$ for all $\eta \in (0, \bar{\eta})$, $\theta'(\bar{\eta}) = 0$ and $\theta'(\eta) < 0$ for all $\eta \in (\bar{\eta}, \infty)$.

It follows from the above that there exists a maximal interval (η_1, ∞) such that $\theta'(\eta) \neq 0$ on (η_1, ∞). Now, suppose that $p\theta'(\eta) > 0$ for all $\eta \in (\eta_1, \infty)$. Then by (11) $(f^m)''(\eta) < 0$ for all $\eta \in (\eta_1, \infty)$. However, by Lemma 3 there exists a point $\eta_2 \in (\eta_1, \infty)$ such that $(f^m)'(\eta_2) < 0$. Whence

$$(f^m)'(\eta) < (f^m)'(\eta_2) < 0 \qquad \text{for all} \quad \eta \in (\eta_2, \infty),$$

but this contradicts (7). We conclude that $p\theta'(\eta) < 0$ on (η_1, ∞), from which the lemma follows.

To complete the proof of Theorem 2 we shall now distinguish between the two cases (i) $q < 0$ and $2p + q > 0$, and (ii) $q > 0$ and $p < 0$. We shall treat each of these cases separately below.

(i) *The Case $q < 0$ and $2p + q > 0$*

It follows from Lemma 5 that $\theta(\eta) \to V$ as $\eta \to \infty$ for some $V \geqslant 0$. Hence to prove the theorem it remains to show that by necessity $V > 0$.

By Lemmas 3 and 5 there exists a point $\bar{\eta} \in (0, \infty)$ such that $(f^m)'(\eta) < 0$ and $\theta'(\eta) < 0$ for all $\eta > \bar{\eta}$. Hence, observing that $(2 - \lambda) = (2p + q)/p > 0$, if one multiplies (11) by η and integrates from $\eta_1 > \bar{\eta}$ to $\eta_2 > \eta_1$ one obtains

$$\begin{aligned} &\eta_1(f^m)'(\eta_1) - f^m(\eta_1) \\ &\quad = \eta_2(f^m)'(\eta_2) - f^m(\eta_2) + p\int_{\eta_1}^{\eta_2} \zeta^{2-\lambda}\theta'(\zeta)\,d\zeta \\ &\quad < p\eta_1^{2-\lambda}\int_{\eta_1}^{\eta_2} \theta'(\zeta)\,d\zeta \\ &\quad = p\eta_1^{2-\lambda}\theta(\eta_2) - p\eta_1^2 f(\eta_1). \end{aligned} \tag{13}$$

Letting $\eta_2 \to \infty$ in (13) we deduce that

$$\eta(f^m)'(\eta) - f^m(\eta) \leqslant p\eta^{2-\lambda}V - p\eta^2 f(\eta) \qquad \text{for all} \quad \eta > \bar{\eta}. \tag{14}$$

Now suppose that $V = 0$. Then, multiplying (14) by $\eta^{(1-2m)/m}/f(\eta)$ and integrating from $\bar{\eta}$ to an arbitrary $\eta > \bar{\eta}$ it holds that

$$\begin{aligned}&\frac{m}{(m-1)}\eta^{(1-m)/m} f^{m-1}(\eta)\\ &\quad\leqslant \frac{m}{(m-1)}\bar{\eta}^{(1-m)/m} f^{m-1}(\bar{\eta}) - \frac{pm}{m+1}\{\eta^{(m+1)/m} - \bar{\eta}^{(m+1)/m}\}.\end{aligned}$$

However, this contradicts (7). We conclude that any solution of Type B satisfies (10) for some $V > 0$.

(ii) *The Case* $q > 0$ *and* $p < 0$

To prove the theorem in this case, we multiply (11) by $\eta^{\lambda-1}$ and integrate from $\eta_1 > 0$ to $\eta_2 > \eta_1$ deriving

$$\begin{aligned}-p\theta(\eta_2) &= -p\theta(\eta_1) + \eta_2^{\lambda-1}(f^m)'(\eta_2) - \eta_1^{\lambda-1}(f^m)'(\eta_1)\\ &\quad - (\lambda-1)\int_{\eta_1}^{\eta_2} \zeta^{\lambda-2}(f^m)'(\zeta)\,d\zeta\\ &= -p\theta(\eta_1) + \eta_2^{\lambda-1}(f^m)'(\eta_2) - \eta_1^{\lambda-1}(f^m)'(\eta_1)\\ &\quad - \frac{m}{m-1}(\lambda-1)\int_{\eta_1}^{\eta_2} \zeta^{-2}\theta(\zeta)(f^{m-1})'(\zeta)\,d\zeta.\end{aligned}$$

In view of the monotonicity of f and θ described in Lemmas 3 and 5, this means that

$$\begin{aligned}-p\theta(\eta_2) &< -p\theta(\eta_1) - \eta_1^{\lambda-1}(f^m)'(\eta_1)\\ &\quad - \frac{m}{m-1}|\lambda-1|\,\eta_1^{-2}\theta(\eta_2)\int_{\eta_1}^{\eta_2}(f^{m-1})'(\zeta)\,d\zeta\\ &< -p\theta(\eta_1) - \eta_1^{\lambda-1}(f^m)'(\eta_1) + \frac{m}{m-1}|\lambda-1|\,\eta_1^{-2}\theta(\eta_2)\,U^{m-1}\end{aligned}$$

for all $\eta_2 > \eta_1 > 0$. Hence if we choose η_1 so large that

$$\frac{m}{m-1}|\lambda-1|\,U^{m-1}\eta_1^{-2} < \frac{1}{2}p$$

it holds that

$$-\tfrac{1}{2}p\theta(\eta_2) < -p\theta(\eta_1) - \eta_1^{\lambda-1}(f^m)'(\eta_1)$$

for all $\eta_2 > \eta_1$. In short this means that $\theta(\eta)$ is bounded on $(0, \infty)$. However, since by Lemma 5, $\theta(\eta)$ is positive and monotonic increasing on $(0, \infty)$, this implies that there exists a $V \in (0, \infty)$ such that $\theta(\eta) \to V$ as $\eta \to \infty$. This completes the proof of Theorem 2.

Theorem 2 identifies the asymptotic behaviour as $\eta \to \infty$ of a solution of Type B. One might enquire however as to the rate of convergence toward the asymptotic limit. The following theorem provides an estimate.

THEOREM 3. *Let $f(\eta)$ be a solution of Type* B *and set* $V = \lim_{\eta\to\infty} \eta^\lambda f(\eta) > 0$.

(i) *If $q = 0$ and $p > 0$ then*

$$f(\eta) - V = O(\beta \operatorname{erfc}(\sigma\eta)) \qquad \textit{as} \quad \eta \to \infty,$$

where

$$\sigma = (\tfrac{1}{2}pV^{1-m}/m)^{1/2}.$$

(ii) *If $q = 0$ and $p \leqslant 0$ then*

$$f(\eta) = V \qquad \textit{for all} \quad \eta \in (0, \infty).$$

(iii) *If $q \neq 0$ then*

$$\eta^\lambda f(\eta) - V = O(\eta^{-(m-1)\lambda-2}) \qquad \textit{as} \quad \eta \to \infty.$$

Proof. (i) We begin by following an argument of Shampine [20]. Note that if $U = 0$ then, by necessity, $\beta > 0$. Thus the integral

$$\int_0^\eta \zeta f^{1-m}(\zeta)\, d\zeta$$

is well defined for $\eta > 0$. Hence we may multiply (4) by

$$\exp\left\{\frac{p}{m}\int_0^\eta \zeta f^{1-m}(\zeta)\, d\zeta\right\}$$

and integrate from 0 to $\eta > 0$. This yields

$$(f^m)'(\eta) = \beta \exp\left\{-\frac{p}{m}\int_0^\eta \zeta f^{1-m}(\zeta)\, d\zeta\right\}. \tag{15}$$

Whence

$$V^m - f^m(\eta) = \beta \int_\eta^\infty \exp\left\{-\frac{p}{m}\int_0^\zeta \xi f^{1-m}(\xi)\, d\xi\right\} d\zeta \tag{16}$$

for all $\eta > 0$. Now, by Lemma 3, $f(\eta) \leqslant W = \max\{U, V\}$ for all $\eta > 0$. Hence, substituting in(16),

$$\begin{aligned} |V^m - f^m(\eta)| &\leqslant |\beta| \int_\eta^\infty \exp\left\{-\frac{p}{2m} W^{1-m}\zeta^2\right\} d\zeta \\ &= \tfrac{1}{2}\,|\beta|\,\pi^{1/2}\operatorname{erfc}(\gamma\eta)/\gamma \end{aligned} \tag{17}$$

for all $\eta > 0$, where $\gamma = (\frac{1}{2} p W^{1-m}/m)^{1/2}$.

Inequality (17) already provides an estimate of the convergence of $f(\eta)$ toward V as $\eta \to \infty$. However, following Peletier [19] we may use (17) to step up the above analysis and sharpen the result. We shall assume that $\beta \neq 0$ since the trivial case $\beta = 0$ is already covered by (17).

It follows from (17) that the integral

$$I = \int_0^\infty \zeta\{f^{1-m}(\zeta) - V^{1-m}\}\, d\zeta$$

is well defined and finite, i.e., $-\infty < I < \infty$. Thus, multiplying (15) by $\exp\{\frac{1}{2} pV^{1-m}\eta^2/m\}$ it is a consequence that

$$(f^m)'(\eta)\exp\{\tfrac{1}{2} pV^{1-m}\eta^2/m\} \to C \qquad \text{as} \quad \eta \to \infty, \tag{18}$$

where $C = \beta \exp\{-pI/m\}$. Rewriting (18) gives

$$f'(\eta) \sim (1/m)\, V^{1-m} C \exp\{-\sigma^2\eta^2\} \qquad \text{as} \quad \eta \to \infty.$$

Hence, by integration

$$V - f(\eta) \sim \frac{\pi^{1/2} V^{1-m}}{2m\sigma}\, C \operatorname{erfc}(\sigma\eta) \qquad \text{as} \quad \eta \to \infty. \tag{19}$$

(ii) This is just a restatement of Lemma 3.

(iii) Set $\theta(\eta) = \eta^\lambda f(\eta)$ for $0 < \eta < \infty$, and choose a point $\eta_0 \in (0, \infty)$ so large that

$$p\theta'(\eta) < 0 \qquad \text{for all} \quad \eta > \eta_0; \tag{20}$$

such a point exists by Lemma 5. Then multiplying (12) by $\eta^2(\theta^m)'(\eta)$ and integrating from arbitrary $\eta_1 > \eta_0$ to $\eta_2 > \eta_1$ yields

$$\begin{aligned}\tfrac{1}{2}\eta_2^2\{(\theta^m)'(\eta_2)\}^2 = \tfrac{1}{2}\eta_1^2\{(\theta^m)'(\eta_1)\}^2 - \tfrac{1}{2}m\lambda(m\lambda+1)\,\theta^{2m}(\eta_2) \\ + \tfrac{1}{2}m\lambda(m\lambda+1)\,\theta^{2m}(\eta_1) + \int_{\eta_1}^{\eta_2} \zeta(\theta^m)'(\zeta)\,\theta'(\zeta) \\ \times \{-p\zeta^{2+(m-1)\lambda} + m(2m\lambda+1)\,\theta^{m-1}(\zeta)\}\,d\zeta. \qquad (21)\end{aligned}$$

Hence, if $\eta_1 > \eta_0$ is chosen so large that

$$|p|\,\eta_1^{2+(m-1)\lambda} > m(2m\lambda+1) \sup_{0<\zeta<\infty} \{\theta^{m-1}(\zeta)\}$$

it is clear that the integral on the right-hand side of (21) is monotonic with respect to η_2. Furthermore $\theta^{2m}(\eta) \to V^{2m}$ as $\eta \to \infty$. Thus $\lim_{\eta\to\infty} \eta^2\{(\theta^m)'(\eta)\}^2$ exists. Suppose though that

$$\eta^2\{(\theta^m)'(\eta)\}^2 > \delta^2 \qquad \text{for all} \quad \eta > \bar{\eta};$$

some $\delta > 0$, $\bar{\eta} > \eta_0$. Then, noting (20),

$$-p(\theta^m)'(\eta) > |p|\,\delta/\eta \qquad \text{for all} \quad \eta > \bar{\eta}.$$

Whereupon, integrating with respect to η,

$$-p\theta^m(\eta) + p\theta^m(\bar{\eta}) > |p|\,\delta \log(\eta/\bar{\eta}) \qquad \text{for all} \quad \eta > \bar{\eta},$$

which contradicts the fact that $\theta(\eta) \to V$ as $\eta \to \infty$. We conclude that

$$\eta^2\{(\theta^m)'(\eta)\}^2 \to 0 \qquad \text{as} \quad \eta \to \infty.$$

However, since

$$(f^m)'(\eta) = \eta^{-m\lambda}(\theta^m)'(\eta) - m\lambda\eta^{-m\lambda-1}\theta^m(\eta),$$

this means that

$$(f^m)'(\eta) \sim -m\lambda V^m \eta^{-1-m\lambda} \qquad \text{as} \quad \eta \to \infty. \qquad (22)$$

Now, noting (22), multiplying (11) by $\eta^{\lambda-1}$ and integrating yields

$$p\theta(\eta) - pV = -\eta^{\lambda-1}(f^m)'(\eta) - (\lambda-1)\int_{\eta}^{\infty} \zeta^{\lambda-2}(f^m)'(\zeta)\,d\zeta \qquad (23)$$

for all $\eta > 0$. Hence, if we substitute (22) into (23) it may be concluded that

$$p\theta(\eta) - pV \sim m\lambda V^m(m\lambda+1)\{2+(m-1)\lambda\}^{-1}\,\eta^{-2-(m-1)\lambda} \qquad \text{as} \quad \eta \to \infty. \qquad (24)$$

COROLLARY. (i) *If $q=0$ and $p>0$ then either*

$$(f^m)'(\eta)=0 \qquad \text{for all} \quad \eta>0,$$

or there exists a constant $C \neq 0$ such that

$$(f^m)'(\eta) \sim C\exp(-\sigma^2\eta^2) \qquad \text{as} \quad \eta \to \infty.$$

(ii) *If $q=0$ and $p \leqslant 0$ then*

$$(f^m)'(\eta)=0 \qquad \text{for all} \quad \eta>0.$$

(iii) *If $q \neq 0$ then*

$$(f^m)'(\eta) \sim -m\lambda V^m \eta^{-1-m\lambda} \qquad \text{as} \quad \eta \to \infty.$$

Remark. It follows from (17), (19), and (24) that the estimates in Theorem 3 are sharp.

5. THE DEPENDENCE OF THE ASYMPTOTIC LIMIT ON THE INITIAL PARAMETERS

In this final section we shall study the dependence of the asymptotic limit as $\eta \to \infty$ of solutions of Type B on the initial parameters U and β in (5). We shall deal first with the simpler case, namely, $q>0$ and $p<0$, or $q=0$ and $p \leqslant 0$, in the following theorem.

THEOREM 4. *Suppose $q>0$ and $p<0$, or $q=0$ and $p \leqslant 0$. For $U \geqslant 0$ let $f(\eta; U)$ denote the unique weak solution of* (4) *satisfying $f(0)=U$ and set*

$$V(U)=\lim_{\eta \to \infty} \eta^\lambda f(\eta; U).$$

Then $V(1)>0$ and

$$V(U)=U^{1+(m-1)\lambda/2}V(1) \qquad \text{for} \quad 0 \leqslant U < \infty.$$

Proof. See Theorems 1 and 2 and the remark following Proposition 1.

In the remaining case, when $q \leqslant 0$ and $2p+q>0$, the situation is not so straightforward; for here given any $U \geqslant 0$ there exists a one parameter family of weak solutions of (4) satisfying $f(0)=U$. We shall nonetheless establish a relationship between the asymptotic limit of a weak solution of (4) and the initial parameters U and β in (5). We begin with some groundwork.

Henceforth we shall assume that $q \leqslant 0$ and $2p+q>0$ and set

$\mathscr{S} = \mathscr{S}_A \cup \mathscr{S}_B$. For $(U, \beta) \in \mathscr{S}$ let $f(\eta; U, \beta)$ denote the unique weak solution of (4) satisfying (5) and set

$$V(U,\beta) = \lim_{\eta \to \infty} \eta^{\lambda} f(\eta; U, \beta).$$

We make the preliminary observation, based on the remark following Proposition 1, that for any $\sigma \geqslant 0$

$$V(\sigma U, \sigma^{(m+1)/2}\beta) = \sigma^{1+(m-1)\lambda/2} V(U,\beta). \tag{25}$$

We now turn to an investigation of the monotonicity of the function $V(U,\beta)$. A basic tool in this investigation will be the following "maximum principle" which extends a similar result proved in [12].

LEMMA 6. *Let $f_1(\eta)$ and $f_2(\eta)$ be two distinct solutions of problem* (4), (5) *in an interval* $(0, a)$ *for some* $a > 0$. *Then there exists at most one point* $\eta_0 \in [0, a)$ *such that* $f_1(\eta_0) = f_2(\eta_0)$.

Proof. Suppose to the contrary that there exist two points in $[0, a)$ where f_1 and f_2 are equal. Then, by continuity and by uniqueness in Lemma 1, without any loss of generality we can designate the points $\eta_1, \eta_2 \in [0, a)$, $\eta_1 < \eta_2$, with the properties

$$f_1(\eta_1) = f_2(\eta_1), \qquad f_1(\eta_2) = f_2(\eta_2),$$
$$f_1(\eta) > f_2(\eta) \qquad \text{for all} \quad \eta \in (\eta_1, \eta_2),$$

and

$$(f_1^m)'(\eta_1) > (f_2^m)'(\eta_1), \qquad (f_1^m)'(\eta_2) < (f_2^m)'(\eta_2).$$

However, multiplying (4) for both f_1 and f_2 by η and integrating from η_1 to η_2 we have

$$\eta_2(f_1^m - f_2^m)'(\eta_2) - \eta_1(f_1^m - f_2^m)'(\eta_1) = (2p+q)\int_{\eta_1}^{\eta_2} \zeta\{f_1(\zeta) - f_2(\zeta)\}\, d\zeta,$$

which is contradictory.

The following result, which can claim some intrinsic interest, is an important consequence of Lemma 6.

PROPOSITION 2. *Suppose $q \leqslant 0$ and $2p + q > 0$.*

(i) *Let (U, β_1), $(U, \beta_2) \in \mathscr{S}_B$ and suppose that $\beta_1 > \beta_2$. Then*

$$f(\eta; U, \beta_1) > f(\eta; U, \beta_2) \qquad \textit{for all} \quad \eta \in (0, \infty).$$

(ii) *Let* (U_1, β), $(U_2, \beta) \in \mathscr{S}_B$ *and suppose that* $U_1 > U_2$. *Then*

(a) *if* $p + q \geqslant 0$

$$f(\eta; U_1, \beta) > f(\eta; U_2, \beta) \qquad \textit{for all} \quad \eta \in [0, \infty);$$

(b) *if* $p + q < 0$ *there exists a point* $\bar{\eta} \in (0, \infty)$ *such that*

$$f(\eta; U_1, \beta) > f(\eta; U_2, \beta) \qquad \textit{for all} \quad \eta \in [0, \bar{\eta}),$$
$$f(\bar{\eta}; U_1, \beta) = f(\bar{\eta}; U_2, \beta),$$

and

$$f(\eta; U_1, \beta) < f(\eta; U_2, \beta) \qquad \textit{for all} \quad \eta \in (\bar{\eta}, \infty).$$

Proof. (i) This follows immediately from Lemma 6.

(ii) For convenience, set $f_i(\eta) = f(\eta; U_i, \beta)$ for $i = 1, 2$. The proof then hinges on the identity

$$(f_i^m)'(\eta) + p\eta f_i(\eta) = \beta + (p + q) \int_0^\eta f_i(\zeta)\, d\zeta \tag{26}$$

for $i = 1, 2$, obtained by integrating (4).

(a) For $p + q \geqslant 0$. Suppose that the assertion is false. Then by continuity there exists a point $\bar{\eta} \in (0, \infty)$ such that $f_1(\eta) > f_2(\eta)$ for all $\eta \in [0, \bar{\eta})$ and $f_1(\bar{\eta}) = f_2(\bar{\eta})$. Moreover, by the uniqueness in Lemma 1, $(f_1^m)'(\bar{\eta}) < (f_2^m)'(\bar{\eta})$. However, from (26) we have

$$(f_1^m - f_2^m)'(\bar{\eta}) = (p + q) \int_0^{\bar{\eta}} (f_1 - f_2)(\zeta)\, d\zeta \geqslant 0,$$

which provides a contradiction.

(b) For $p + q < 0$. In this case from (26) we deduce that

$$\begin{aligned} (f_1^m - f_2^m)'(\eta) &+ p\eta^{1-\lambda}\{\eta^\lambda f_1(\eta) - \eta^\lambda f_2(\eta)\} \\ &= (p + q) \int_0^\eta (f_1 - f_2)(\zeta)\, d\zeta \end{aligned} \tag{27}$$

for all $\eta > 0$. Note that $1 - \lambda = (p + q)/p < 0$ and that by (7), $(f_i^m)'(\eta) \to 0$ as $\eta \to \infty$ for $i = 1, 2$. Thus we may let $\eta \to \infty$ in (27) deriving

$$\int_0^\infty (f_1 - f_2)(\zeta)\, d\zeta = 0. \tag{28}$$

Now by Lemma 6, either $f_1(\eta) > f_2(\eta)$ for all $\eta \in [0, \infty)$, or there exists a point $\bar{\eta} \in (0, \infty)$ such that $f_1(\eta) > f_2(\eta)$ for all $\eta \in [0, \bar{\eta})$, $f_1(\bar{\eta}) = f_2(\bar{\eta})$, and $f_1(\eta) < f_2(\eta)$ for all $\eta \in (\bar{\eta}, \infty)$. By (28), though, the latter option is the only possibility. Thus the proposition is proved.

Remark. It can easily be shown that Proposition 2 part (i) remains true for all (U, β_1), $(U, \beta_2) \in \mathscr{S}$, and that Proposition 2 part (ii) remains true for all (U_1, β), $(U_2, \beta) \in \mathscr{S}$ when $p + q \neq 0$. However, $(U, 0) \in \mathscr{S}_A$ for all $U \geqslant 0$ when $p + q = 0$.

We are now in a position to describe the monotonicity of $V(U, \beta)$ in $\mathscr{S}$.

PROPOSITION 3. (i) *Let* (U, β_1), $(U, \beta_2) \in \mathscr{S}$ *and suppose that* $\beta_1 > \beta_2$. *Then* $V(U, \beta_1) > V(U, \beta_2)$.

(ii) *Let* (U_1, β), $(U_2, \beta) \in \mathscr{S}$ *and suppose that* $U_1 > U_2$. *Then*

(a) *if* $p + q > 0$, $V(U_1, \beta) > V(U_2, \beta)$,

(b) *if* $p + q = 0$, $V(U_1, \beta) = V(U_2, \beta)$,

(c) *if* $p + q < 0$, $V(U_1, \beta) < V(U_2, \beta)$.

Proof. Noting Theorem 1, since $V(U, \beta) > 0$ for $(U, \beta) \in \mathscr{S}_B$ and $V(U, \beta) = 0$ for $(U, \beta) \in \mathscr{S}_A$, to prove the proposition it suffices to consider only initial data in $\mathscr{S}_B$.

(i) Since the result has already been established for $q = 0$ [12] we shall confine our attention to the case $q < 0$ (and $\lambda > 0$). For convenience we set $f_i(\eta) = f(\eta; U, \beta_i)$ and $V_i = V(U, \beta_i)$ for $i = 1, 2$.

Multiplying (4) by η for both f_1 and f_2, integrating and subtracting yields

$$\eta(f_1^m - f_2^m)'(\eta) - (f_1^m - f_2^m)(\eta) + p\eta^2(f_1 - f_2)(\eta)$$
$$= (2p + q) \int_0^\eta \zeta(f_1 - f_2)(\zeta)\, d\zeta$$

for all $\eta > 0$. Consequently, by Theorem 3 and its corollary,

$$(2p + q) \int_0^\eta \zeta(f_1 - f_2)(\zeta)\, d\zeta$$
$$= p\eta^{2-\lambda}(V_1 - V_2) + O(\eta^{-m\lambda}) \qquad \text{as} \quad \eta \to \infty. \tag{29}$$

Now, by Proposition 2, $V_1 \geqslant V_2$. If we suppose though that $V_1 = V_2$, then by (29)

$$\int_0^\infty \zeta(f_1 - f_2)(\zeta)\, d\zeta = 0,$$

which contradicts Proposition 2.

(ii) For convenience set $f_i(\eta) = f(\eta; U_i, \beta)$ and $V_i = V(U_i, \beta)$ for $i = 1, 2$.

(a) For $p + q > 0$. Integrating (4) for both f_1 and f_2 and subtracting yields

$$(f_1^m - f_2^m)'(\eta) + p\eta(f_1 - f_2)(\eta) = (p+q)\int_0^\eta (f_1 - f_2)(\zeta)\, d\zeta$$

for all $\eta > 0$. Thus by Theorem 3 and its corollary

$$(p+q)\int_0^\eta (f_1 - f_2)(\zeta)\, d\zeta = p\eta^{1-\lambda}(V_1 - V_2) + O(\eta^{-1-m\lambda}) \qquad \text{as} \quad \eta \to \infty.$$

We proceed now along lines of the proof of part (i). By Proposition 2, $V_1 \geqslant V_2$. However, if we suppose that $V_1 = V_2$ we obtain, in this instance,

$$\int_0^\infty (f_1 - f_2)(\zeta)\, d\zeta = 0,$$

which contradicts Proposition 2.

(b) For $p + q = 0$. In this case direct integration of (4) yields

$$-\beta + pV(U, \beta) = 0$$

for any $(U, \beta) \in \mathscr{S}_B$.

(c) For $p + q < 0$. In this final case by Proposition 2 there exists a point $\bar{\eta} \in (0, \infty)$ such that $f_1(\bar{\eta}) = f_2(\bar{\eta})$ and $f_1(\eta) < f_2(\eta)$ for all $\eta \in (\bar{\eta}, \infty)$. Constructing integrations over the interval $(\bar{\eta}, \infty)$ the proof may be completed identically to that of part (i). We omit further details.

We now show that $V(U, \beta)$ is continuous in $\mathscr{S}$. This will be done in three steps. First we show that $V(U, \beta)$ is continuous in $\mathscr{S}_B$, next in $\mathscr{S} \backslash \{(0, 0)\}$, and finally in the whole of $\mathscr{S}$. Each of these steps will be represented by a lemma below. Throughout, for $(U_0, \beta_0) \in \mathscr{S}$ and $\delta > 0$ we shall set

$$\mathscr{B}_\delta(U_0, \beta_0) = \{(U, \beta) \in \mathscr{S}_B : (U - U_0)^2 + (\beta - \beta_0)^2 < \delta^2\}.$$

LEMMA 7. *$V(U, \beta)$ is continuous in $\mathscr{S}_B$.*

Proof. Pick $(U_0, \beta_0) \in \mathscr{S}_B$ and $\delta > 0$. We begin by showing that there exists a $(U^*, \beta^*) \in \mathscr{S}_B$ such that

$$f(\eta; U, \beta) < f(\eta; U^*, \beta^*) \qquad \text{for all} \quad \eta \in (0, \infty) \tag{30}$$

and for all $(U, \beta) \in \mathscr{B}_\delta(U_0, \beta_0)$.

By Proposition 3; when $p+q \geqslant 0$ it holds that $V(U,\beta) < V(U_0+\delta, \beta_0+\delta)$ for all $(U,\beta) \in \mathscr{B}_\delta(U_0,\beta_0)$ and when $p+q<0$ it holds that $V(U,\beta) < V(0,\beta_0+\delta)$ for all $(U,\beta) \in \mathscr{B}_\delta(U_0,\beta_0)$. In either case $\bar{V} = \sup\{V(U,\beta) : (U,\beta) \in \mathscr{B}_\delta(U_0,\beta_0)\} < \infty$. Next choose $\beta_1 > 0$ such that $(1,\beta_1) \in \mathscr{S}_B$ and choose a U^* so large that $U^* > U_0+\delta$ and $(U^*)^{1+(m-1)\lambda/2} V(1,\beta_1) > \bar{V}$. We assert that (U^*,β^*), where $\beta^* = (U^*)^{(m+1)/2}\beta_1$, has the desired property.

Let $(U,\beta) \in \mathscr{B}_\delta(U_0,\beta_0)$ be arbitrary. Then

$$U < U_0 + \delta < U^*,$$

and hence (30) holds for small values of η, while by (25),

$$V(U,\beta) \leqslant \bar{V} < (U^*)^{1+(m-1)\lambda/2} V(1,\beta_1) = V(U^*,\beta^*),$$

and therefore (30) holds for large values of η. It follows from Lemma 6 that (30) must then hold for all values of $\eta \in (0,\infty)$. Thus (U^*,β^*) has the desired property.

Next, multiplying (11) for any $(U,\beta) \in \mathscr{S}_B$ by $\eta^{\lambda-1}$ and integrating we derive

$$V(U,\beta) - F(\eta; U,\beta) = (1-\lambda)\, I(\eta; U,\beta),$$

where

$$F(\eta; U,\beta) = \eta^\lambda f(\eta; U,\beta) + \frac{1}{p}\eta^{\lambda-1}(f^m)'(\eta; U,\beta) + \frac{(1-\lambda)}{p}\eta^{\lambda-2} f^m(\eta; U,\beta)$$

and

$$I(\eta; U,\beta) = \frac{(2-\lambda)}{p}\int_\eta^\infty \zeta^{\lambda-3} f^m(\zeta; U,\beta)\, d\zeta.$$

Hence for all $(U,\beta) \in \mathscr{B}_\delta(U_0,\beta_0)$,

$$\begin{aligned} |V(U,\beta) - V(U_0,\beta_0)| &\leqslant |F(\eta; U,\beta) - F(\eta; U_0,\beta_0)| + |1-\lambda|\,\{I(\eta; U,\beta) + I(\eta; U_0,\beta_0)\} \\ &\leqslant |F(\eta; U,\beta) - F(\eta; U_0,\beta_0)| + 2\,|1-\lambda|\, I(\eta; U^*,\beta^*) \end{aligned}$$

for all $\eta \in (0,\infty)$.

Now, given $\varepsilon > 0$ we may pick an η^* so large that

$$2\,|1-\lambda|\, I(\eta^*, U^*,\beta^*) < \tfrac{1}{2}\varepsilon.$$

Whereupon, by the continuous dependence of positive classical solutions of (4) on initial data recalled from Lemma 1, we can pick a $\delta_0 \in (0, \delta)$ such that

$$|F(\eta^*; U, \beta) - F(\eta^*; U_0, \beta_0)| < \tfrac{1}{2}\varepsilon$$

for all $(U, \beta) \in \mathscr{B}_{\delta_0}(U_0, \beta_0)$. It follows that

$$|V(U, \beta) - V(U_0, \beta_0)| < \varepsilon \qquad \text{for all} \quad (U, \beta) \in \mathscr{B}_{\delta_0}(U_0, \beta_0).$$

Since $(U_0, \beta_0) \in \mathscr{S}_B$, $\delta > 0$, and $\varepsilon > 0$ were arbitrary this completes the proof of Lemma 7.

LEMMA 8. *Suppose $U_0 > 0$ and $(U_0, \beta_0) \in \mathscr{S}_A$. Then given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $V(U, \beta) < \varepsilon$ for all $(U, \beta) \in \mathscr{B}_\delta(U_0, \beta_0)$.*

Proof. Fix $\varepsilon > 0$. Set $a = \sup\{\eta \in [0, \infty) : f(\eta; U_0, \beta_0) > 0\}$, $0 < a < \infty$, and $\theta_0(\eta) = \eta^\lambda f(\eta; U_0, \beta_0)$ for $0 \leqslant \eta \leqslant \infty$. Then since $\theta_0(\eta) \to 0$ as $\eta \to a-$, there exists a point $a_1 \in (0, a)$ such that

$$0 < \theta_0(a_1) = a_1^\lambda f(a_1; U_0, \beta_0) < \tfrac{1}{2}\varepsilon$$

and

$$\theta_0'(a_1) = a_1^\lambda f'(a_1; U_0, \beta_0) + \lambda a_1^{\lambda-1} f(a_1; U_0, \beta_0) < 0.$$

Now, since by Lemma 1 positive classical solutions of (4) depend continuously on initial data, there exists a $\delta > 0$ such that for all $(U, \beta) \in \mathscr{B}_\delta(U_0, \beta_0)$

$$0 < a_1^\lambda f(a_1; U, \beta) < \tfrac{1}{2}\varepsilon$$

and

$$a_1^\lambda f'(a_1; U, \beta) + \lambda a_1^{\lambda-1} f(a_1; U, \beta) < 0.$$

Hence, if for arbitrary $(U, \beta) \in \mathscr{B}_\delta(U_0, \beta_0)$ we set $\theta(\eta) = \eta^\lambda f(\eta; U, \beta)$ it holds that $\theta(a_1) < \tfrac{1}{2}\varepsilon$ and $\theta'(a_1) < 0$. However, by Lemma 5, this means that $\theta(\eta) < \tfrac{1}{2}\varepsilon$ for all $\eta \in (a_1, \infty)$. Hence $V(U, \beta) < \varepsilon$. In other words there exists a $\delta > 0$ such that $V(U, \beta) < \varepsilon$ for all $(U, \beta) \in \mathscr{B}_\delta(U_0, \beta_0)$.

LEMMA 9. *Given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $V(U, \beta) < \varepsilon$ for all $(U, \beta) \in \mathscr{B}_\delta(0, 0)$.*

Proof. (i) If $p + q > 0$. Then by Proposition 3

$$V(U, \beta) < V(\delta, \delta) < V(\delta^{2/(m+1)}, \delta) \qquad \text{for all} \quad (U, \beta) \in \mathscr{B}_\delta(0, 0)$$

and $0 < \delta < 1$. Hence, by (25)

$$V(U,\beta) < \delta^{[2+(m-1)\lambda]/(m+1)}V(1, 1) \qquad \text{for all} \quad (U,\beta) \in \mathscr{B}_\delta(0,0)$$

and $0 < \delta < 1$.

(ii) If $p+q \leqslant 0$. Then by Proposition 3

$$V(U,\beta) < V(0,\delta) \qquad \text{for all} \quad (U,\beta) \in \mathscr{B}_\delta(0,0),$$

but by (25)

$$V(0,\delta) = \delta^{[2+(m-1)\lambda]/(m+1)}V(0, 1).$$

Observing that $V(U,\beta) = 0$ for all $(U,\beta) \in \mathscr{S}_A$ and thus that trivially $V(U,\beta)$ is continuous in $\mathscr{S}_A$, combining Lemmas 7, 8, and 9 produces the promised result.

PROPOSITION 4. *$V(U,\beta)$ is continuous in $\mathscr{S}$.*

To complete our investigation of the function $V(U,\beta)$ we shall study its range.

For fixed $U \in [0,\infty)$ let

$$\bar{V}_\beta(U) = \sup\{V(U,\beta) : (U,\beta) \in \mathscr{S}\}.$$

Then we assert that $\bar{V}_\beta(U) = \infty$ for all $U \geqslant 0$. When $U = 0$, by (25)

$$V(0,\beta) = \beta^{[2+(m-1)\lambda]/(m+1)}V(0, 1) \qquad \text{for all} \quad \beta \geqslant 0,$$

so plainly $\bar{V}_\beta(0) = \infty$. Suppose though that $\bar{V}_\beta(U_0) < \infty$ for some $U_0 > 0$. Then by (25)

$$\bar{V}_\beta(U) = (U/U_0)^{1+(m-1)\lambda/2}\bar{V}_\beta(U_0) < \infty \qquad \text{for all} \quad U > 0.$$

However, in view of the fact that $\bar{V}_\beta(0) = \infty$, this contradicts the continuity of $V(U,\beta)$ in $\mathscr{S}_B$. We conclude that indeed $\bar{V}_\beta(U) = \infty$ for all $0 \leqslant U < \infty$.

Next we observe that for $p+q \leqslant 0$, for fixed $\beta \geqslant 0$ the range of $V(U,\beta)$ is prescribed by Theorem 1 and Proposition 3. If, however, $p+q > 0$ the situation is not so clear. In this case for fixed $\beta \in (-\infty,\infty)$, let

$$\bar{V}_U(\beta) = \sup\{V(U,\beta) : (U,\beta) \in \mathscr{S}\}.$$

Plainly, by (25), $\bar{V}_U(0) = \infty$. We assert though that $\bar{V}_U(\beta) = \infty$ for all $\beta \in (-\infty,\infty)$. For supposing to the contrary that $\bar{V}_U(\beta_0) < \infty$ for some $\beta_0 \neq 0$, then by (25)

$$\bar{V}_U(\beta) = (\beta/\beta_0)^{[2+(m-1)\lambda]/(m+1)}\bar{V}_U(\beta_0) < \infty \qquad \text{for all} \quad \beta/\beta_0 > 0.$$

However, in view of the continuity of $V(U,\beta)$ in $\mathscr{S}_B$ this contradicts $\bar{V}_U(0)=\infty$.

From a summary of the results of this section the following theorem has evolved.

THEOREM 5. *Suppose $q \leqslant 0$ and $2p+q>0$. For $(U,\beta) \in \mathscr{S} = \mathscr{S}_A \cup \mathscr{S}_B$ let $f(\eta; U,\beta)$ denote the weak solution of problem* (4), (5) *and set* $V(U,\beta) = \lim_{\eta\to\infty} \eta^\lambda f(\eta; U,\beta)$.

(i) *If $p+q>0$ then $V(U,\beta)$ is continuous and strictly monotonic increasing with respect to U and β in $\mathscr{S}$. Furthermore for fixed $U \geqslant 0$, $V(U,\beta) \to \infty$ as $\beta \to \infty$, and for fixed $\beta \in (-\infty,\infty)$, $V(U,\beta)\to\infty$ as $U\to\infty$.*

(ii) *If $p+q=0$ then $V(U,\beta)=\beta/p$ for all $(U,\beta)\in\mathscr{S}$.*

(iii) *If $p+q<0$ then $V(U,\beta)$ is continuous, strictly monotonic decreasing with respect to U, and strictly monotonic increasing with respect to β in $\mathscr{S}$. Furthermore for fixed $U\geqslant 0$, $V(U,\beta)\to\infty$ as $\beta\to\infty$.*

As a consequence of Theorem 5 we are able to state the following theorem, which extends a similar result confined to the case $q=0$ proved in [12].

THEOREM 6. *Suppose $q \leqslant 0$ and $2p+q>0$. For $(U,\beta) \in \mathscr{S} = \mathscr{S}_A \cup \mathscr{S}_B$ let $f(\eta; U,\beta)$ denote the weak solution of problem* (4), (5) *and set* $V(U,\beta) = \lim_{\eta\to\infty} \eta^\lambda f(\eta; U,\beta)$.

(i) *Then given any $U\geqslant 0$ and $V^*\geqslant 0$ there exists a unique $\beta\in(-\infty,\infty)$ such that $(U,\beta)\in\mathscr{S}$ and $V(U,\beta)=V^*$.*

(ii) (a) *If $p+q>0$, then given any $\beta\in(-\infty,\infty)$ and $V^*\geqslant 0$ there exists a unique $U\geqslant 0$ such that $(U,\beta)\in\mathscr{S}$ and $V(U,\beta)=V^*$ if and only if $\beta\leqslant 0$, or $\beta>0$ and $V^*\geqslant(0,\beta)$.*

(b) *If $p+q=0$, then given any $\beta\in(-\infty,\infty)$ and $V^*\geqslant 0$ there exists a $U\geqslant 0$ such that $(U,\beta)\in\mathscr{S}$ and $V(U,\beta)=V^*$ if and only if $\beta\geqslant 0$ and $V^*=\beta/p$ in which case $V(U,\beta)=V^*$ for all $U\geqslant 0$.*

(c) *If $p+q<0$, then given any $\beta\in(-\infty,\infty)$ and $V^*\geqslant 0$ there exists a unique $U\geqslant 0$ such that $(U,\beta)\in\mathscr{S}$ and $V(U,\beta)=V^*$ if and only if $\beta\geqslant 0$ and $V^*\leqslant V(0,\beta)$.*

REFERENCES

1. W. F. AMES, Similarity for the nonlinear diffusion equation, *Ind. Engrg. Chem. Fundam.* **4** (1965), 72–76.
2. W. F. AMES, "Nonlinear Partial Differential Equations in Engineering," Academic Press, New York, 1965.

3. F. V. ATKINSON AND L. A. PELETIER, Similarity profiles of flows through porous media, *Arch. Rational Mech. Anal.* **42** (1971), 369–379.
4. F. V. ATKINSON AND L. A. PELETIER, Similarity solutions of the nonlinear diffusion equation, *Arch. Rational Mech. Anal.* **54** (1974), 373–392.
5. G. I. BARENBLATT, On some unsteady motions of a liquid and a gas in a porous medium, *Prikl. Mat. Meh.* **16** (1952), 67–78.
6. G. I. BARENBLATT, On a class of exact solutions of the plane one-dimensional problem of unsteady filtration into a porous medium, *Prikl. Mat. Meh.* **17** (1953), 739–742.
7. G. I. BARENBLATT, On limiting self-similar motions in the theory of unsteady filtration of a gas in a porous medium and the theory of the boundary layer, *Prikl. Mat. Meh.* **18** (1954), 409–414.
8. G. I. BARENBLATT AND YA. B. ZEL'DOVICH, On the dipole-type solution in problems of unsteady gas filtration in the polytropic regime, *Prikl. Mat. Meh.* **21** (1957), 718–720.
9. A. H. CRAVEN AND L. A. PELETIER, Similarity solutions for degenerate quasilinear parabolic equations, *J. Math. Anal. Appl.* **38** (1972), 73–81.
10. C. J. VAN DUYN AND L. A. PELETIER, A class of similarity solutions of the nonlinear diffusion equation, *Nonlinear Anal.* **1** (1977), 223–233.
11. B. H. GILDING AND L. A. PELETIER, On a class of similarity solutions of the porous media equation, *J. Math. Anal. Appl.* **55** (1976), 351–364.
12. B. H. GILDING AND L. A. PELETIER, On a class of similarity solutions of the porous media equation II, *J. Math. Anal. Appl.* **57** (1977), 522–538.
13. P. HARTMAN, "Ordinary Differential Equations," Philip Hartman, Baltimore, Maryland, 1973.
14. A. S. KALASHNIKOV, The occurrence of singularities in solutions of the non-steady seepage equation, *Zh. Vychisl. Mat. i Mat. Fiz.* **7** (1967), 440–444; English transl., *USSR Comput. Math. and Math. Phys.* **7** (1967), 269–275.
15. C. F. LEE, On the existence of solutions of a nonlinear differential diffusion system, *Proc. Roy. Soc. Edinburgh Sect. A* **71** (1972), 1–7.
16. R. E. MARSHAK, Effect of radiation on shock wave behavior, *Phys. Fluids* **1** (1958), 24–29.
17. O. A. OLEINIK, A.S. KALASHNIKOV, AND C. YUI-LIN, The Cauchy problem and boundary problems for equations of the type of unsteady filtration, *Izv. Akad. Nauk SSSR Ser. Mat.* **22** (1958), 667–704.
18. R. E. PATTLE, Diffusion from an instantaneous point source with a concentration-dependent coefficient, *Quart. J. Mech. Appl. Math.* **12** (1959), 407–409.
19. L. A. PELETIER, Asymptotic behaviour of temperature profiles of a class of non-linear heat conduction problems, *Quart. J. Mech. Appl. Math.* **23** (1970), 441–447.
20. L. F. SHAMPINE, Concentration-dependent diffusion, *Quart. Appl. Math.* **30** (1973), 441–452.
21. L. F. SHAMPINE, Concentration-dependent diffusion. II. Singular problems, *Quart. Appl. Math.* **31** (1973), 287–293.
22. L. F. SHAMPINE, Concentration-dependent diffusion. III. An approximate solution, *Quart. Appl. Math.* **33** (1976), 429–431.
23. K. K. TAM, A note on the similarity solutions for degenerate quasilinear parabolic equations. *J. Math. Anal. Appl.* **45** (1974), 132–135.
24. YA. B. ZEL'DOVICH AND A. S. KOMPANEEC, On the theory of propagation of heat with the heat conductivity depending upon the temperature, in "Collection Commemorating the Seventieth Birthday of Academician A.F. Ioffe," pp. 61–72, Izdat. Akad. Nauk SSSR, Moscow, 1950.
25. YA. B. ZEL'DOVICH AND YU. P. RAIZER, "Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, Vol. II," Academic Press, New York, 1967.